# NONLINEAR WAVES ON A VERTICAL FLUID FILM

A. P. Ivanskii

UDC 532.516

In this paper we consider waves of small, but finite, amplitude, generated on the surface layer of a viscous incompressible fluid running on a vertical solid surface. The treatment is for small Reynolds numbers $\mathrm{Re} = U_0h_0/\nu$, and is based on the full system of Navier-Stokes equations with boundary conditions on the wall and on the free surface of the fluid (the action of surface tension is taken into account).

A similar problem, but with no surface tension, and for not too large inclination angles of the wall, was considered in [1, 2].

An equation of the form

$$h_t + 3h_x + 6hh_x + \alpha h_{xx} + \gamma h_{xxxx} = 0 \tag{1}$$

is used in [3, 4], taking into account effects of nonlinearity, "negative viscosity" (the term $\alpha h_{xx}$), and dissipation (the term $\gamma h_{xxxx}$) for waves on the surface of the fluid film.

A similar equation was obtained in [5] for concentration waves in chemically reacting diffusion media. Also given is an analytic solution of this equation in the stationary case, as well as qualitative discussion of the mechanisms of pumping, dissipation, and energy transfer as obtained from the spectra described by this equation. Numerical analysis of the perturbation evolution was carried out in [6] for Eq. (1) with periodic boundary conditions, and it has been shown that during the evolution process the regular initial perturbation reaches a turbulent state (chemical turbulence).

As already mentioned, Eq. (1) describes nonlinearity, energy transfer, and dissipation for waves in active media, but, unfortunately, does not include dispersion effects. In the given paper we use an equation of the form

$$h_t + 3h_x + 6hh_x + \alpha h_{xx} + \beta h_{xxx} + \gamma h_{xxxx} = 0$$

for waves on the surface of a fluid film for small Reynolds numbers. It obviously is the simplest form of a wave equation including all effects enumerated above, especially dispersion ($\beta \neq 0$). It is also attempted to study several properties of its solution. A fuller study of the behavior of the solution must, obviously, be carried out by numerical analysis.

The full system of equations with boundary conditions, describing the flow of a fluid film over a vertical wall is, in dimensionless variables, is

$$u_t + uu_x + vu_y + p_x = \left(u_{xx} + \frac{u_{yy}}{\delta^2}\right)\Big/\mathrm{R} + \frac{Lg}{U_0^2}, \tag{2}$$

$$\delta^2(v_t + uv_x + vv_y) + p_y = \delta^2\left(v_{xx} + \frac{yy}{\delta^2}\right)\Big/\mathrm{Re},$$

$$u_x + v_y = 0.$$

The boundary conditions are:

$$u = v = 0|_{y=0},$$

$$v = h_t + uh_x|_{y=h}, \tag{3}$$

$$p = 2\left[v_y + \delta^2h_x^2u_x - h_x(u_y + \delta^2v_x)\right]/\mathrm{Re}\,(1 + \delta^2h_x^2) - \delta^2\mathrm{W}h_{xx}\,{}_{|y=h},$$

$$(u_y + \delta^2v_x)(1 - \delta^2h_x^2) - 2\delta^2h_x(u_x - v_y) = 0|_{y=}\,,^*$$

where u, v are the vertical and horizontal components of the dimensionless velocity, p is pressure, x and y are the coordinates along and across the film, and nondimensionalizing is performed as follows: $x = x'/L$, $y = y'/h_0$, $u = u'/U_0$, $v = Lv'/h_0U_0$, $p = (P' - P_a')/\rho U_0^2$, $h = h'/h_0$, where L is a characteristic longitudinal dimension, $h_0$ is the unperturbed width of the film, $U_0 = gh_0^2/\nu$ is a characteristic flow velocity, $\nu$ is viscosity, and the following

*As in Russian original — Publisher.

Novosibrisk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 52-58, March-April, 1980. Original article submitted April 26, 1979.

0021-8944/80/2102-0201$07.50 

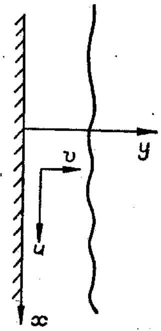


Fig. 1

parameters were introduced: $\delta = h_0/L$, characterizing the wavelength, $\mathrm{Re} = U_0L/\nu$ is the Reynolds number, $W = \sigma/\rho h_0 U_0^2$ is the Weber number, characterizing the surface tension, $\sigma$ is the surface tension, and g is the gravitational acceleration.

We represent the solution of this system in the form of a series in a small parameter $\varepsilon \ll 1$, the deviation of the surface from the unperturbed position:

$$
\begin{aligned}
h &= 1 + \varepsilon\eta, \\
u &= U + \varepsilon u_1 + \varepsilon^2 u_2, \\
v &= \varepsilon v_1 + \varepsilon^2 v_2, \\
p &= p_l + \varepsilon p_1 + \varepsilon^2 p_2.
\end{aligned}
\tag{4}
$$

Consider the case $\mathrm{Re} \sim 1$, $\delta \ll 1$ (wavelength). Substituting expressions (4) into (2), (3) and equating terms of identical powers in $\varepsilon$, we obtain in the zeroth order

$$
U_{yy}/\mathrm{Re}\,\delta^2 + Lg/U_0^2 = p_{lx}, \quad p_{ly} = 0,
$$
$$
U_y = p_l = 0|_{y=1}, \quad U = 0|_{y=0},
$$

whence follow $p_l = 0$ and $U = y - y^2/2$, i.e., the ordinary Poiseuille profile.

Transforming to the following two approximations, we choose the relation between the wavelength, and the amplitude in the form $\delta^2 = k\varepsilon$, where k is a number of order 1.

This assumption implies that the velocity of the wave dispersion is of the same order as the velocity of its nonlinear torsion. Under these conditions the system of equations acquires the form

$$
\begin{aligned}
&\varepsilon(u_{1t} + Uu_{1x} + U_y v_1 + p_{1x}) = [\varepsilon u_{1xx} + (u_{1yy} + \varepsilon u_{2yy})/k]/\mathrm{Re}, \\
&\varepsilon p_{1y} + \varepsilon^2 p_{2y} = k\varepsilon[\varepsilon v_{1xx} + (v_{1yy} + \varepsilon v_{2yy})/k]/\mathrm{Re}, \\
&\varepsilon(u_{1x} + v_{1y}) + \varepsilon^2(u_{2x} + v_{2y}) = 0, \\
&u_1 = u_2 = v_1 = v_2 = 0|_{y=0}, \\
&\varepsilon v_1 + \varepsilon^2 v_2 = \varepsilon\eta_t + (U + \varepsilon u_1)\varepsilon\eta_x|_{y=1+\varepsilon\eta}, \\
&\varepsilon p_1 + \varepsilon^2 p_2 = 2(\varepsilon v_{1y} + \varepsilon^2 v_{2y} - \varepsilon\eta_x U_y - \varepsilon^2\eta_x u_{1y})/\mathrm{Re} - k\varepsilon^2 W\eta_{xx}|_{y=1+\varepsilon\eta}, \\
&\varepsilon U_{yy}\eta + \varepsilon u_{1y} + \varepsilon^2 u_{2y} + k\varepsilon^2 v_{1x} - 2k\varepsilon^3\eta_x(u_{1x} - v_{1y}) = 0|_{y=1+\varepsilon\eta}.
\end{aligned}
\tag{5}
$$

Retaining everywhere the dominant terms in $\varepsilon$, we obtain the following approximation:

$$
\begin{aligned}
&u_{1yy} = 0, \quad p_{1y} = v_{1yy}/\mathrm{Re}, \\
&u_{1x} + v_{1y} = 0, \\
&u_1 = v_1 = 0|_{y=0}, \quad v_1 = \eta_t + U\eta_x|_{y=1}, \\
&p_1 = 2v_{1y}/\mathrm{Re} - \delta^2 W\eta_{xx}|_{y=1}, \\
&U_{yy}\eta + u_{1y} = 0|_{y=1}.
\end{aligned}
$$

This system of equations has a solution in form of the so-called "kinematic" waves [7]:

$$
\begin{aligned}
&u_1 = y\eta, \quad v_1 = -\eta_x y^2/2, \\
&\eta = \eta(x - t), \quad p_1 = -\eta_x(y + 1)/\mathrm{Re} - \delta^2 W\eta_{xx}.
\end{aligned}
\tag{6}
$$

As seen from Eq. (6), in this approximation they are stationary and their velocity is $c = gh_0^2/\nu$.

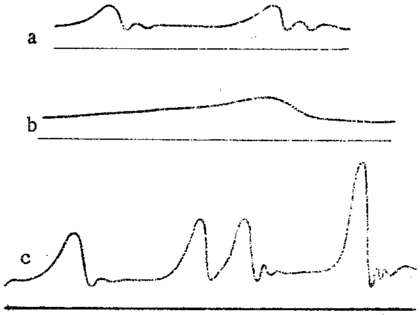

Fig. 2

To obtain an evolution equation of the shape of the surface $\eta(x, t)$ it is necessary to construct the following approximation, in which one assumes

$$\eta = \eta(\xi, \tau), \tag{7}$$

where $\xi = x - t$; $\tau = \varepsilon t$.

Substituting (7) into (5) and transferring the boundary conditions to the point $y = 1$, we obtain

$$(U - 1)u_{1\xi} + U_y v_1 + p_{1\xi} = u_{1\xi\xi}/\mathrm{Re} + u_{2yy}/\mathrm{Re}k,$$
$$u_{2\xi} + v_{2y} = 0,\ u_2 = v_2 = 0|_{y=0},$$
$$-v_{1y}\eta + v_2 = \eta_\tau + u_1\eta_\xi\ |_{y=1},$$
$$u_{2y} + kv_{1\xi} = 0|_{y=1}.$$

Substituting the known $u_1$, $v_1$, $p_1$ and eliminating the unknown $u_2$, $v_2$ from the kinematic boundary condition we obtain an equation for $\eta$:

$$\eta_\tau + 2\eta\eta_\xi + (2/15)\,\mathrm{Re}\,k\eta_{\xi\xi} + k\eta_{\xi\xi\xi} + \left(\frac{\mathrm{Re}\,k\delta^2 \mathrm{W}}{3}\right)\eta_{\xi\xi\xi\xi} = 0,$$

or in dimensional form

$$h_t + 2U_0[(h-h_0)/h_0]h_\xi + (2/15)U_0\left(\frac{gh_0^4}{\nu^2}\right)h_{\xi\xi} + U_0 h_0^2 h_{\xi\xi\xi} + \left(\frac{U_0\sigma h_0}{3\rho g}\right)h_{\xi\xi\xi\xi} = 0,$$

where all quantities are dimensional.

This equation is an evolution equation describing nonlinear waves of small, but finite, amplitude on the surface of a vertical fluid film. The term with $h_{\xi\xi}$ describes energy transfer into the wave from the stationary basic flow, the term with $h_{\xi\xi\xi\xi}$ the action of surface tension, and the term with $h_{\xi\xi\xi}$ "hydrodynamic" dispersion of waves. The presence of this term in the equation, as seen below, can importantly affect the behavior of the solution (e.g., wave seclusion).

For the analysis we rewrite the equation in the form

$$\varphi_t + 2\varphi\varphi_\xi + \frac{2}{15}\frac{h_0}{L}\mathrm{R}\varphi_{\xi\xi} + \frac{h_0^2}{L^2}\varphi_{\xi\xi\xi} + \frac{\sigma h_0}{3\rho g L^3}\varphi_{\xi\xi\xi\xi} = 0, \tag{8}$$

where $\varphi = (h' - h_0')/h_0'$; $t = U_0 t'/L$; $R = gh_0^3/\nu^2$.

Consider a solution of Eq. (8) in the form of stationary waves. We seek a solution in the form $\varphi = \varphi(\zeta) = \varphi(\xi - VT)$; then

$$-V\varphi' + 2\varphi\varphi' + \frac{2}{15}\frac{h_0}{L}\mathrm{R}\varphi'' + \frac{h_0^2}{L^2}\varphi''' + \frac{\sigma h_0}{3\rho g L^3}\varphi'''' = 0. \tag{9}$$

Introducing the film number $\mathrm{Fi} = \sigma^3/\rho^3 g\nu^4$, and integrating (9) over $\zeta$ from $-\infty$ to $\infty$ with account taken of the fact that for a solitary wave of the soliton type $\varphi = \varphi' = \varphi'' = \varphi''' = \varphi'''' = 0$ for $\zeta \to \pm\infty$, while for a "step" $\varphi = 0$, $\zeta \to +\infty$, $\varphi = V$ for $\zeta \to -\infty$, we obtain

$$(\varphi - V)\,\varphi + \frac{2}{15}\left(\frac{h_0}{L}\right)\mathrm{R}\varphi' + \left(\frac{h_0}{L}\right)^2\varphi'' + \frac{1}{3}\frac{\mathrm{Fi}^{1/3}h_0^3}{\mathrm{R}^{2/3}L^3}\varphi''' = 0. \tag{10}$$

Equation (10) makes it possible to analyze several characteristic features of waves, observed in film flow. In Fig. 2 we show several typical profiles of film width. It is seen that for these natural waves (Fig. 2C)

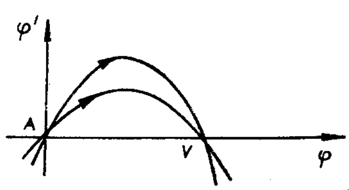


Fig. 3

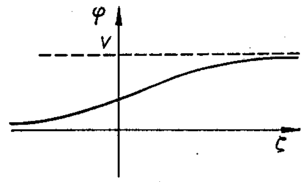


Fig. 4

one observes a strong spread in wavelength. This can be explained qualitatively by Eq. (10) as follows. In the region of small width the main terms of the equation are $(\varphi - V)\varphi$ and $\frac{2}{15}\frac{h_0}{L}R\varphi'$, so it acquires the form

$$(\varphi - V)\varphi + \frac{2}{15}\frac{h_0}{L}R\varphi' = 0.$$

The phase trajectories of this equation have the shape of Fig. 3, where the arrow corresponds to increasing $\zeta = \xi - Vt$. When $\varphi$ approaches the point $\varphi = V$ the derivative $\varphi' \to 0$, and this approximation takes place asymptotically for $\zeta \to +\infty$, i.e., the solution in this region has the shape of Fig. 4.

In an experimental situation, however, a small random perturbation can lead to a vanishing $\varphi'$ and to phase points merging again into position A. Since the perturbation is random in nature, the wavelength is also random. The shape of the surface in the return region to point A is determined by terms with high derivatives. In the case of wave excitation a definite wavelength is extracted, corresponding to the excitation frequency.

The effect of the dispersion term and of the term involving surface tension on the shape of solitary waves and "steps" can be analyzed by considering the behavior of phase trajectories of Eq. (10) in neighborhoods of stationary points for $\zeta \to \pm\infty$.

Consider the case of solitons. Substituting into the equation

$$\varphi \sim e^{\varkappa\zeta/\delta},$$

we obtain the characteristic equation

$$\varkappa^3 + a\varkappa^2 + b\varkappa - c = 0, \tag{11}$$

where $a = 3R^{2/3}/Fi^{1/3}$; $b = (6/15)R^{5/3}/Fi^{1/3}$; $c = 3VR^{2/3}/Fi^{1/3}$.

The behavior of solutions near stationary points is determined by whether Eq. (11) has real or complex roots. The effect of the dissipation term on the presence of oscillations at the edges for $\zeta \to \pm\infty$ is of interest, since in the case $a = 0$, $b > 0$ the criterion of having three real roots [8]

$$A = \frac{27(3b - a^2)^3}{(2a^3 - 9ab - 27c)^2} < -6.75 \tag{12}$$

is not satisfied, and, consequently, we always have one real and two complex conjugate roots $\varkappa_1$, $\varkappa_{2,3} = \alpha \pm i\omega$.

From the Vieta theorem it follows that

$$\varkappa_1\varkappa_2\varkappa_3 = \varkappa_1(\alpha^2 + \omega^2) = \frac{3VR^{2/3}}{Fi^{1/3}},$$

$$\varkappa_1\varkappa_2 + \varkappa_2\varkappa_3 + \varkappa_1\varkappa_3 = b,$$

$$\varkappa_1 + \varkappa_2 + \varkappa_3 = \varkappa_1 + 2\alpha = -a,$$

i.e., for $V > 0$, $a = 0$ we obtain $\varkappa_1 > 0$ and $\alpha = -\varkappa_1/2 < 0$, i.e., for $\zeta \to +\infty$ the bounded solution $e^{\varkappa\zeta/\delta}$ corresponds to the complex roots, and $\zeta \to -\infty$ to the real one. Thus, a soliton has the shape of Fig. 5a, i.e., in front the solution is oscillating, and behind it is smooth. The situation is contrary for $V < 0$. Waves of this type were obtained numerically [9] and observed experimentally [10].

We show now the possibility of soliton existence without oscillations when the term with a third derivative is included. For this it is necessary that the condition $A < -6.75$ be satisfied (the presence of three real roots), while the roots must have different signs.

This is possible, e.g., for the case $a = 0.3$, $b = 0.004$, $c = 0.003$ corresponding to the values $\varkappa_1 = 0.08$, $\varkappa_2 = -0.16$, $\varkappa_3 = -0.234$, $R = 0.1$, $Fi = 10$, $V = 0.01$, or $a = 3$, $b = 1.25$, $c = 1.5$, corresponding to the values $\varkappa_1 = 0.5$, $\varkappa_2 = -1.5$, $\varkappa_3 = -2.0$, $R = 3.125$, $Fi = 10$, $V = 0.5$.

In this case the soliton has the shape illustrated in Fig. 5b. To explain the general conditions of absence of oscillations we use the fact that, as is well known from algebra (see, e.g., [11]), all roots of Eq. (11) are located

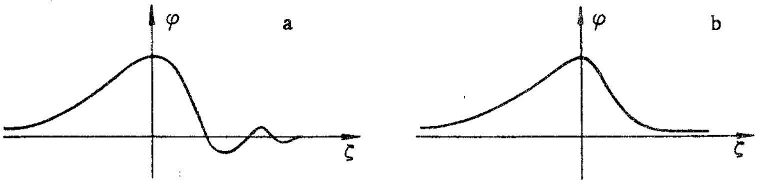


Fig. 5

in the left complex half-plane if and only if all its coefficients, i.e., $a$, b, and −c, have the same sign, and, besides

$$-c - ba < 0.$$

Thus, for V > 0 the roots $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ are either all positive or have different signs. The first possibility cannot be realized, since both extrema points of the cubic parabola (11) are always in the left half-plane for $a > 0$ and b > 0. Consequently, a soliton without oscillations can exist if condition (12) is satisfied and V > 0, or for V < 0

$$3(-V)R^{2/3}/Fi^{1/3} - ba \geqslant 0.$$

The author is grateful to V. E. Nakoryakov for guidance, and to V. G. Gasenko and O. Yu. Tsvelodub for useful remarks and advice.

## LITERATURE CITED

1. R. S. Johnson, "Shallow water waves on a viscous fluid. The undular bore," Phys. Fluids, 15, No. 10 (1972).
2. Yu. P. Ivanilov, "Rolling waves in an inclined channel," Zh. Vychisl. Mat. Fiz., 1, No. 6 (1961).
3. A. A. Nepomnyashchii, "Stability of wave regimes in a film flowing on an inclined plane," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 3 (1974).
4. L. N. Maurin, G. É. Odishariya, and A. A. Tochigin, "A solitary wave on a flowing fluid film," in: S. S. Kutateladze (editor), Nonlinear Wave Processes in Two-Phase Media [in Russian], Inst. Teplofiz. Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1977).
5. Y. Kuramoto and T. Tsuzuki, "Persistent propagation of concentration waves in dissipative media far from thermal equilibrium," Prog. Theor. Phys., 55, No. 2 (1976).
6. T. Yamada and Y. Kuramoto, "A reduced model showing chemical turbulence," Prog. Theor. Phys., 56, No. 2 (1976).
7. G. R. Whitham, Linear and Nonlinear Waves, Wiley, New York (1974).
8. B. M. Shumyagskii, Tables for Solving Cubic Equations [in Russian], Gostekhizdat, Moscow (1950).
9. O. Yu. Tsvelodub, "Stationary plane waves on a flowing fluid film," in: Thermophysical Studies [in Russian], Inst. Teplofiz. Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1977).
10. B. G. Pokusaev and S. V. Alekseenko, "Two-dimensional waves on a vertical fluid film," in: Nonlinear Wave Processes in Two-Phase Media [in Russian], Inst. Teplofiz. Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1977).
11. M. A. Lavrent'ev and B. V. Shabat, Methods of the Theory of Functions of Complex Variables [in Russian], Nauka, Moscow (1973).